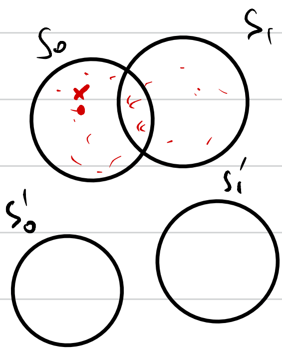
Metric Spaces and Topology
Lecture 11

Disjointifying sets. Given a family (set) $\mathcal{F}$ of sets, which are not necessarily disjoint, there is a way to make them disjoint without changing the sets much. More formally, we replace each set $S \in \mathcal{F}$ with the set $\pi(S) = S' := \{(x, S) : x \in S\}$, let $\mathcal{F}' := \{S' : S \in \mathcal{F}\}$. Then $\forall\, S_0, S_1 \in \mathcal{F}$, $S_0' \cap S_1' = \emptyset$.



$S_0 \mapsto S_0'$

$x \mapsto (x, S_0)$

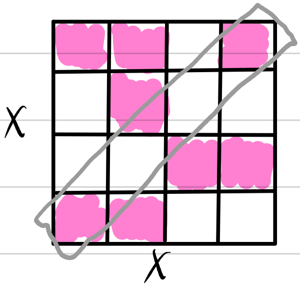
Given a choice function c' for $\mathcal{F}'$, i.e. $c' : \mathcal{F}' \to \bigcup \mathcal{F}'$, we can define a choice function

$$c := \mathrm{proj}_1 \circ c' \circ \pi : \mathcal{F} \to \bigcup \mathcal{F}$$

$$S \mapsto S' \mapsto \underset{\substack{\parallel \\ (x,S)}}{c'(S')} \mapsto x \in S.$$

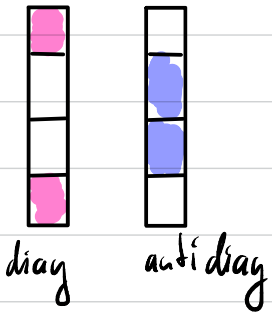
Continuum. We start by proving that there exist unctbl sets.
By Cantor diagonalization method, we show that for any set S, $\mathcal{P}(S) \not\equiv S$, where $\mathcal{P}(S) := \{S' : S' \subseteq S\}$.
E.g. $S = \emptyset \Rightarrow \mathcal{P}(S) = \{\emptyset\}$. If $S \equiv n$, then $\mathcal{P}(S) \equiv 2^n$.
The Cantor diagonalization method is a basic

algorithm of producing a new column vector in an $n \times n$ table:



Want a vector

that doesn't appear in the table. We take the antidiagonal:

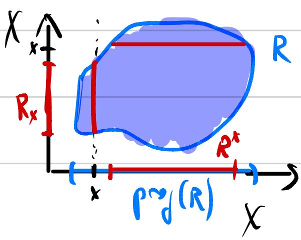


One checks that if antidiag is not equal to column k because its k^{th} coordinate is different from the (k,k) entry of the table.

More generally, given a set X and $R \subseteq X^2$ (binary relation on X), we define its antidiagonal

$$AD(R) := \{x \in X : (x,x) \notin R\}.$$

For $x \in X$, the vertical (resp. horizontal) fiber or section of R is the set $R_x := \{y \in X : (x,y) \in R\}$ (resp. $R^x := \{y \in X : (y,x) \in R\}$).



Cantor's Diagonalization Theorem. $\forall X$ & $R \subseteq X^2$, $AD(R)$ is not a vertical or horizontal section of R.

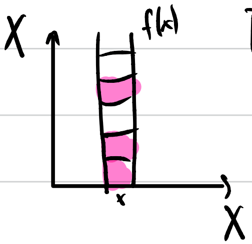
Proof. $\forall x \in X$, $x \in AD(R) :\iff (x,x) \notin R \iff x \notin R_x$
$\iff x \notin R^x$

Thus, $AD(R) \neq R_x$ for all $x \in X$.

Same for R^x. $\square$

Cantor's theorem. For any set X, $X \not\twoheadrightarrow \mathcal{P}(X)$ (equiv. $\mathcal{P}(X) \not\hookrightarrow X$). In particular, $\mathcal{P}(X) \not\equiv X$.

Proof. Suppose $\exists\, f : X \to \mathcal{P}(X)$ surjective. We define
$$R := \{(x,y) : y \in f(x)\}.$$



Thus, $AD(R) \subseteq X$ is not be equal to any vertical fiber $f(x)$. Thus, f isn't surjective, a contradiction. $\square$

Thus, $2^X \equiv \mathcal{P}(X)$ is strictly "bigger" than X.
$$\begin{array}{ccc} 2^X & \equiv & \mathcal{P}(X) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1_Y & \longmapsfrom & Y \end{array}$$
Obviously, $X \hookrightarrow \mathcal{P}(X)$.
$x \mapsto \{x\}$

In particular, $2^{\mathbb{N}} \equiv \mathcal{P}(\mathbb{N})$ is unctbl. We know that $2^{\mathbb{N}} \equiv \mathcal{C} \subseteq [0,1]$ but also $2^{\mathbb{N}} \twoheadrightarrow [0,1]$ via binary representation, so $[0,1] \hookrightarrow 2^{\mathbb{N}}$, hence $2^{\mathbb{N}} \hookrightarrow [0,1] \hookrightarrow 2^{\mathbb{N}}$. Thus, by the following theorem, $2^{\mathbb{N}} \equiv [0,1]$. Hence also, $2^{\mathbb{N}} \equiv \mathbb{R}$.

Cantor–Schröder–Bernstein Theorem. If $A \hookrightarrow B$ and $B \hookrightarrow A$, then $A \equiv B$.

Proof. $\boxed{\text{HW}}$.

The equinumerosity class of $\mathcal{P}(\mathbb{N})$ is called *continuum*, and any set equinumerous to $\mathcal{P}(\mathbb{N})$ is said to have *cardinality continuum*.

Example. $2^{\mathbb{N}} \equiv \mathbb{N}^{\mathbb{N}}$.

Proof. $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ so it's enough to prove $\mathbb{N}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$. Use the unary representation $n \mapsto \underbrace{11\ldots 1}_{n \text{ times}}$ and 0s in between: $(3, 5, 4, \ldots) \mapsto (1110\,11111\,0\,1111\,0\ldots)$. $\square$

$\boxed{\text{HW}}$ Find the image of this injection.

Let's do an application. A metric space is called *separable* if it admits a ctbl dense set.

Prop. Any separable metric space X has cardinality at most continuum, in fact $Q^{\mathbb{N}} \twoheadrightarrow X$ for any dense subset $Q \subseteq X$.

Proof. Let $Q \subseteq X$ be a ctbl dense subset of X. We fix $x_0 \in X$, and define $f: Q^{\mathbb{N}} \to X$

$$(q_n) \mapsto \begin{cases} \lim_n q_n & \text{if this exists.} \\ x_0 & \text{otherwise} \end{cases}$$

f is surjective because for any $x \in X$, there is a sequence (q_n) converging to x (defined by <u>choosing</u> q_n from $B_{\frac{1}{n}}(x)$). $\square$

It's natural to wonder, and Cantor did, if there is an unctbl set smaller than continuum (i.e. $\hookrightarrow$ but no bijection), in other words, is there an unctbl subset of $\mathbb{R}$ not equinumerous to $\mathbb{R}$? The negative answer is known as:

<u>Continuum Hypothesis (CH).</u> There is no unctbl set that $\hookrightarrow \mathbb{R}$ but $\not\approx \mathbb{R}$.

CH was shown by K. Gödel to be consistent with ZFC and then it was prove P. Cohen (using a new method called forcing) that $\neg$ CH is also consistent with ZFC.